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Creep and relaxation Poisson's ratio: Back to the foundations of linear viscoelasticity. Application to concrete

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ABSTRACT

Various scientific communities, mainly in concrete and polymer materials fields, have extended the definition of the Poisson's ratio to linear viscoelasticity. Depending on the authors, the viscoelastic Poisson's ratios can be increasing, decreasing or non-monotonic functions of time. Going back to the classic integral formulation of the linear viscoelastic behaviour, creep and relaxation Poisson's ratios are rederived as functions of bulk and shear relaxation or compliance functions. Both non ageing and ageing behaviours are considered. A literature survey on the thermodynamic restrictions on the viscoelastic characteristics shows that the ageing case has been much less studied than the non ageing case. Still, some examples, both theoretical, including in the ageing case, and practical, regarding concrete, are provided to highlight that any evolution of the viscoelastic Poisson's ratios is possible: increasing, decreasing and even non monotonous.

1. Introduction

The question of the multiaxial behaviour of concrete has been a matter of study at EDF since the works of Granger (1995) who highlighted the need of performing multiaxial tests in order to properly model the biaxial creep occurring in concrete containment buildings of nuclear power plants. This need has led to starting a broad experimental program on multiaxial creep in 2004 at EDF CEIDRE, in France. These tests are described in Charpin et al. (2015); Galenne et al. (2013), Charpin et al. (*To be submitted*). The overall topic of predicting the delayed behaviour still motivates works on the creep Poisson's ratio as the present work performed at EDF R&D and also outside under EDF funding (Aili et al., 2015b, 2015a), through collaborations (Torrenti et al., 2014) or independently (Hilaire, 2014). As a matter of fact, the topic of the creep Poisson's ratio has led to a lot of studies, but their comparison is sometimes difficult due to different definitions used for the various viscoelastic Poisson's ratio (which was already the motivation of works on viscoelastic Poisson's ratios such as Hilton and Yi (1998)). Hence, the present article aims at helping to share knowledge between the concrete science community and the mechanics community, in order to improve the way the multiaxial behaviour of concrete is studied. To introduce the need for such a work, a historical perspective on studies about the multiaxial behaviour of concrete and linear viscoelasticity are proposed successively. Throughout this article, the terms creep Poisson's ratio and relaxation Poisson's ratio will be used extensively and replaced by CPR and RPR.

Apart from the multiaxial nature of concrete creep, its ageing viscoelastic behaviour is also modeled at EDF R&D. The notations chosen in the present paper to detail the equations of viscoelasticity are vastly inspired of these recent works (Sanahuja, 2013; Sanahuja and Huang, 2016).

To further illustrate how the topic of the description of the multiaxial behaviour of concrete through the use of CPRs has been dealt with in the concrete science community, a chronological presentation is used in order to give a historical perspective. The present review of the use of the viscoelastic Poisson's ratio starts with Gopalakrishnan work in 1968. Prior to that a number of works were done on multiaxial creep of concrete, of which a review can be found in Gopalakrishnan's thesis (Gopalakrishnan, 1968). However in this review, little detail is given about the formula used to compute the viscoelastic Poisson's ratio. Therefore these works are not dealt with in detail and the reader is referred to Gopalakrishnan's thesis.

Gopalakrishnan et al. (1969); Gopalakrishnan (1968) have undergone a broad study of multiaxial creep of concrete. In their 1969 article, CPRs are computed using data from multiaxial basic creep tests (in concrete science, the terms *basic* and *autogenous* refer to the fact that moisture exchange between the sample and the surrounding air is prevented). Their aim is to verify if the CPR is the same under different states of stress. The strains used to

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compute the CPR are the basic creep strains, to which the autogeneous shrinkage of a companion specimen are subtracted as well as the elastic strains at loading.

To compute the CPR, the authors first compute the uniaxial compliance $J_u(t)$ using a uniaxial creep test. Then, they assume that during a triaxial test, there might be one different CPR for each direction. This assumption is surprising because in general anisotropic elasticity, each PR is a property of two directions. The CPR in each direction is computed from strains of the triaxial test in this direction and the knowledge of the uniaxial compliance.

The computation of the CPR is hence independent from that of the elastic PR. Using these relations, they show that the CPR depends on the triaxiality of the loading.

Jordaan and Illston also published creep test results on cubic samples in 1969 (Jordaan and Illston, 1969). The strain used is the basic creep strain of which the autogeneous shrinkage is subtracted.

Their expression are completely different to Gopalakrishnan et al.'s. Little justification is given in the paper, but the equations were re-derived by Benboudjema in an internal document at EDF (Benboudjema et al., 2000). Four different expressions are used for the CPR depending on the loading type (uniaxial, biaxial, hydrostatic or general triaxial).

These expressions are complex but rigorous. They allow computing an apparent CPR in case the material is not exactly isotropic.

York et al. (1970), in a study of multiaxial creep on cylindrical samples loaded with an axial stress and a radial stress, computed the CPR by simply using the same relation as for the PR in elasticity, replacing elastic strains by the creep strains corrected of shrinkage strains. To our knowledge, it is the first time this relation is used to compute the CPR. If the material is truly isotropic, it yields the same results as Jordan and Illston's. This approach is also used in the final report of the study (Kennedy, 1975) and in McDonald (1976).

As will be seen in the theoretical presentation in Section 2, this approach is the closest to the one which will be adopted in this paper, based on isotropic linear viscoelasticity.

Parrott performed in 1974 uniaxial tests on cement paste samples during which axial and transverse strains were measured (Parrott, 1974). The CPR is computed using only the creep strain, i.e. the total strain minus the elastic and shrinkage strain. This approach has the inconvenient that it yields a quantity which cannot be easily related to volumetric and deviatoric compliances.

Grasley and Lange (2007) used the RPR. It is simply called the viscous Poisson's ratio, and it is computed in the transformed domain from the bulk and shear relaxation moduli using an equation which is analogous to Eq. (41) of this document. Since the correspondence principle is applicable to the RPR (as will be shown in the sequel), all the usual relations between elastic coefficients can be used in the transformed domain. This definition is rigorous but requires assuming that the material is non ageing in order to use the Laplace transform, and does not permit computing the Poisson's ratio directly in the time domain which makes it more difficult to use for experimentalists.

More recently, Aili et al. (2015b) developed in their article a theory which is consistent with the one summarized in the present paper. However, we here propose a historical perspective on the topic and try to be exhaustive concerning the useful relations that have been proposed in the literature on isotropic ageing and non ageing linear viscoelasticity on the topic of viscoelastic Poisson's ratios.

In their article, Aili et al. define the creep and relaxation Poisson's ratio in the time domain, in the case of ageing linear viscoelasticity, using uniaxial creep and relaxation experiments. The relation (40) between creep and relaxation Poisson's ratio is shown.

A variety of points of view on the viscoelastic Poisson's ratio have been expressed in the literature on concrete science. As a consequence, the equations used to compute the PRs differ from one author to the other. The aim of the present paper is to provide a comprehensive presentation of the relations which can be used to rigorously compute and manipulate viscoelastic PRs and relate them to compliance and relaxation functions. To do so, we propose to use the classical theory of isotropic linear viscoelasticity.

As mentioned by Gurtin and Sternberg (1962), the development of the theory of linear viscoelasticity was initiated by Boltzmann (1878) and Volterra (1909). The elaboration of this mechanical and mathematical theory lead to Volterra's theory of functionals (Volterra, 1959) and made possible subsequent developments in the50's. In the particular case of non ageing viscoelasticity, spectral decomposition of relaxation function was established (Biot, 1954) and the correspondence principle, allowing the use of results established in the case of elasticity for viscoelasticity, using the Fourier transform (Read, 1950; Sips, 1951) or most of the time the Laplace–Carson transform (Mandel, 1955; Lee, 1960; Biot, 1958). An overview of these studies can be found in Mandel (1966) in French, or Christensen (2012).

Ageing viscoelasticity has also been studied, but closed-form solution are more difficult to derive (Mandel, 1958). However, most materials (polymers, concrete) are ageing, which is why in this work we try to mention the existing results in that field.

Thermodynamics has allowed deriving restrictions on the relaxation functions in a non ageing framework (Biot, 1954; Mandel, 1966; Day, 1970; Christensen, 1972, 2012) but results concerning the ageing case are scarce. In the field of concrete science, empirical knowledge about the evolution of relaxation and compliance functions exist (Bažant, 1975; Bažant and Wittmann, 1982; Jiràsek, 2015), but the thermodynamic theory of linear ageing viscoelasticity seems, to the knowledge of the authors, incomplete.

The theory of linear viscoelasticity has been particularized to the isotropic case by many authors (Mandel, 1966; Christensen, 2012), leading to the introduction of bulk and shear compliance and relaxation functions, but also to the rigorous definition of creep and relaxation Poisson's ratios (in ageing and non ageing cases). The RPR has the advantage that in the non ageing case, it can be used along with the correspondence principle. A general equation between the CPR and the RPR has been established (see for example Salençon, 1983), but the origin of this relation is not clear to the present authors. It is valid in the ageing case. It has also been shown (apparently independently) by van der Varst and Kortsmit (1992) in the non ageing case.

Outside concrete science, applications on polymers and even food has motivated the use of viscoelastic Poisson's ratio, with again a variety of approaches (Grassia et al., 2010; Lu et al., 1997; Pandini and Pegoretti, 2011; Ashrafi et al., 2008).

The topic has been recognized as a difficult one and has been the focus of articles by Lakes (1992); Lakes and Wineman (2006), Hilton and Yi (1998); Hilton (2001, 2009, 2011) and other authors (Tschoegl et al., 2002; van der Varst and Kortsmit, 1992). In these articles, the distinction between various kinds of viscoelastic Poisson's ratios is discussed. However we believe that Hilton's classifications are not complete since a clear differentiation between the relaxation and the creep Poisson's ratio is missing. Moreover, a debate about the monotonicity of the relaxation Poisson's ratio has been initiated by Tschoegl et al. (2002); Lakes and Wineman (2006) and further discussed on the basis of experimental data on polymers by Grassia et al. (2010).

Therefore we believe that even if the definitions of the RPR and CPR from the linear viscoelasticity theory are straightforward, a summary of this theory as well as explanations about the significance of these Poisson's ratio and how they relate to other coefficients would be beneficial to the concrete science community, and also to other communities working on viscoelastic materials.

As shown in this introduction, the viscoelastic Poisson's ratios have been used extensively in concrete science, but sometimes not exploiting fully the knowledge that mechanics can bring on these quantities. The aim of this article is to bridge this gap. To do so, first a theoretical presentation of linear viscoelasticity is proposed. The parallels with elasticity are highlighted. Then a synthetic presentation of all useful relations between the compliances, relaxation functions and the viscoelastic Poisson's ratios is proposed. The present authors believe that these formulas are very useful to help dealing correctly with isotropic linear viscoelasticity. Second, discussions about the thermodynamical restrictions on viscoelastic parameters and about the validity and applicability of Poisson's ratios in viscoelasticity are proposed. Finally, four examples of the use of Poisson's ratios in concrete science are proposed. Three of them are computational applications, among which one considers ageing; one is experimental. A final example from the field of polymers is proposed in order to show the wide applicability of the theory reviewed in this article.

2. Material behaviour in linear viscoelasticity

This section introduces the linear viscoelastic behaviour, in the general anisotropic case, and particularized to isotropy. Then the uniaxial creep and relaxation experiments are modelled, to derive the creep and relaxation Poisson ratios. Eventually, useful relations between viscoelastic material parameters are gathered. The ageing and non ageing cases are considered.

2.1. General 3D behaviour

The anisotropic linear viscoelastic behaviour is recalled in this section. Even if these equations can be found in classical textbooks, they serve to introduce the notations adopted in this paper, and for comprehensiveness.

2.1.1. Response to a step of stress or strain

As the linear viscoelastic behaviour is completely defined by the response to a step of stress or strain, the latter is considered first. If the stress is imposed equal to:

$$\sigma(t) = \sigma^0 \mathrm{H}(t - t^0) \tag{1}$$

Where H is the Heaviside function, then the strain response can be written using a creep compliance $\mathbb{J}$, which is a fourth order tensor with minor symmetries, depending on two time variables (t, t^0). The second variable represents the time at which the stress increment is applied, the first one represents the time at which the strain is measured. The consequences of a loading increment are only seen at times larger than the loading time. Hence:

$$\forall\ t < t',\ \ \mathbb{J}(t, t') = 0 \tag{2}$$

Due to the assumption of linear behaviour, the strain response writes:

$$\varepsilon(t) = \mathbb{J}(t, t^0) : \sigma^0 \tag{3}$$

Similarly, if a strain step:

$$\varepsilon(t) = \varepsilon^0 \mathrm{H}(t - t^0) \tag{4}$$

is applied, the stress response can be written using a relaxation tensor $\mathbb{R}$:

$$\sigma(t) = \mathbb{R}(t, t^0) : \varepsilon^0 \tag{5}$$

Let us now generalize these equations to the case of complex stress and strain histories.

2.1.2. Boltzmann formula and Stieltjes integrals

For any strain or stress history, the 3D linear viscoelastic mechanical behaviour can be written using the creep compliance $\mathbb{J}$. As a convenient alternative to Boltzmann integrals, Stieltjes integrals can be used (Volterra, 1959; Mandel, 1966).

$$\varepsilon(t) = \int_{t'=-\infty}^{t} \mathbb{J}(t, t') : \mathrm{d}\sigma(t')$$
$$\text{or} \quad \varepsilon(t) = \mathbb{J}(t, .) \vdots \sigma(.) \text{ or } \varepsilon = \mathbb{J} \vdots \sigma \tag{6}$$

where $\vdots$ is the Volterra integral tensor operator. The Stieltjes integrals will be used in this document because of their very compact form.

Likewise, the material behaviour can be expressed using the relaxation tensor $\mathbb{R}$.

$$\boldsymbol{\sigma}(t) = \int_{t'=-\infty}^{t} \mathbb{R}\left(t, t'\right) : \mathrm{d}\boldsymbol{\varepsilon}\left(t'\right)$$
$$\text{or} \quad \boldsymbol{\sigma}(t) = \mathbb{R}(t, .) \vdots \boldsymbol{\varepsilon}(.) \quad \text{or} \quad \boldsymbol{\sigma} = \mathbb{R} \vdots \boldsymbol{\varepsilon} \tag{7}$$

The relaxation and compliance tensors are inverses in the sense of the Volterra integral tensor operator:

$$\int_{t'=-\infty}^{t} \mathbb{R}(t, t') : \mathrm{d}\mathbb{J}(t', t^0) = \mathrm{H}(t - t^0)\mathbb{I}$$
$$\text{or} \quad \mathbb{R}(t, .) \vdots \mathbb{J}(., t^0) = \mathrm{H}(t - t^0)\mathbb{I}$$
$$\text{or} \quad \mathbb{R} \vdots \mathbb{J} = \mathrm{H}\mathbb{I} \tag{8}$$

2.1.3. 3D non ageing linear viscoelasticity

When the material is non ageing, $\mathbb{J}(t, t') = \mathbb{J}(t - t')$ and $\mathbb{R}(t, t') = \mathbb{R}(t - t')$. Thanks to this assumption, the Boltzmann or Stieltjes integrals are transformed into convolution products. Eq. (6) becomes:

$$\varepsilon(t) = \int_{t'=-\infty}^{t} \mathbb{J}(t - t') : \mathrm{d}\sigma(t') \quad \text{or} \quad \varepsilon = \mathbb{J} \overset{\star}{\vdots} \sigma \tag{9}$$

Eq. (7) becomes:

$$\sigma(t) = \int_{t'=-\infty}^{t} \mathbb{R}(t - t') : \mathrm{d}\varepsilon(t') \quad \text{or} \quad \sigma = \mathbb{J} \overset{\star}{\vdots} \varepsilon \tag{10}$$

Where the notation $\overset{\star}{\vdots}$ is not exactly the usual Riemann convolution product, but its derivative, which is called the Stieltjes convolution product (Mandel, 1966). Now, introducing the Laplace–Carson transform of a time function f:

$$f^{\star}(p) = p \int_{-\infty}^{\infty} f(t) e^{-pt}\, \mathrm{d}t \tag{11}$$

Eqs. (9) and (10) can be written in the transformed domain:

$$\varepsilon^{\star} = \mathbb{J}^{\star} : \sigma^{\star} \quad \text{and} \quad \sigma^{\star} = \mathbb{R}^{\star} : \varepsilon^{\star} \tag{12}$$

While the relation between the creep compliance and the relaxation function becomes

$$\mathbb{J}^{\star} : \mathbb{R}^{\star} = \mathbb{I} \tag{13}$$

2.2. Isotropic linear viscoelasticity

A presentation of this topic was given by Mandel as early as 1958 (Mandel, 1958). The relations presented here are close to Mandel's except analogs of the bulk and shear moduli in elasticity are used instead of Lamé coefficients.

2.2.1. General stress or strain histories

Let us now assume that material is isotropic. The behaviour can now be described by two scalar functions. A convenient way to write the isotropic behaviour is to project the previous relations on the basis $\mathbb{J}, \mathbb{K}$ of isotropic fourth order tensors:

$$\mathbb{R}(t, t') = 3R_k(t, t')\mathbb{J} + 2R_g(t, t')\mathbb{K} \tag{14}$$

introducing the bulk R_k and shear R_g relaxation functions. The scalar factors 3 and 2 are introduced to mimic the expression of the elastic isotropic stiffness tensor as a function of the bulk and shear moduli. The behaviour (7) thus becomes:

$$\sigma = (R_k \circ \operatorname{tr} \varepsilon)1 + 2R_g \circ \varepsilon^{\text{dev}} \tag{15}$$

introducing the Volterra integral operator $\circ$.

The tensor of compliance functions is written similarly:

$$\mathbb{J}(t,t') = \frac{1}{3}J_k(t,t')\mathbb{J} + \frac{1}{2}J_g(t,t')\mathbb{K} \tag{16}$$

introducing the bulk J_k and shear J_g compliance functions. The behaviour (6) becomes:

$$\varepsilon = \frac{1}{9}(J_k \circ \operatorname{tr} \sigma)1 + \frac{1}{2}J_g \circ \sigma^{\text{dev}} \tag{17}$$

According to (8), the bulk (resp. shear) relaxation and compliance functions are inverses in the sense of the Volterra integral operator:

$$R_k \circ J_k = \mathrm{H} \quad \text{and} \quad R_g \circ J_g = \mathrm{H} \tag{18}$$

2.2.2. Case of a uniaxial stress history

As both the uniaxial creep and relaxation tests involve a uniaxial stress tensor history $\sigma(t) = \sigma_{11}(t)\underline{e}_1 \otimes \underline{e}_1$, this specific case is investigated here, without referring to any particular form for the stress history $\sigma_{11}(t)$.

Use of the compliances. Eq. (17) still holds in this particular case, and allows to write the longitudinal (along 11) and transverse (along 22) components of the strain tensor history as:

$$\varepsilon_{11} = \left(\frac{J_k}{9} + \frac{J_g}{3}\right) \circ \sigma_{11} \tag{19}$$

$$\varepsilon_{22} = \left(\frac{J_k}{9} - \frac{J_g}{6}\right) \circ \sigma_{11} \tag{20}$$

The ratio of the longitudinal strain to the longitudinal stress histories can then be written as a function of either the longitudinal stress or the longitudinal strain history:

$$\frac{\varepsilon_{11}}{\sigma_{11}} = \frac{(J_k/9 + J_g/3) \circ \sigma_{11}}{\sigma_{11}} = \frac{\varepsilon_{11}}{(J_k/9 + J_g/3)^{-1} \circ \varepsilon_{11}} \tag{21}$$

where the last expression has been obtained from the inversion of (19), solving the Volterra integral equation for σ_{11}. Be aware that in this text, the fractional bar represents the usual scalar division, whereas the "−1" exponent represent the inversion in the sense of the Volterra integral operator "$\circ$". This is critical to avoid confusions. Both equivalent expressions of (21) will be useful to interpret the particular cases of the creep (where the evolution of σ_{11} is prescribed) and relaxation (where the evolution of ε_{11} is prescribed) tests.

The ratio of the transverse strain to the longitudinal strain histories can be written similarly:

$$\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{(J_k/9 - J_g/6) \circ \sigma_{11}}{(J_k/9 + J_g/3) \circ \sigma_{11}} = \frac{(J_k/9 - J_g/6) \circ (J_k/9 + J_g/3)^{-1} \circ \varepsilon_{11}}{\varepsilon_{11}} \tag{22}$$

Use of the relaxation functions. Now using Eq. (15) in the case of uniaxial stress history, and using the fact that $\varepsilon_{22} = \varepsilon_{33}$ as the material behaviour is isotropic, one gets two relations:

$$\sigma_{11} = \left(R_k + \frac{4}{3}R_g\right) \circ \varepsilon_{11} + \left(2R_k - \frac{4}{3}R_g\right) \circ \varepsilon_{22} \tag{23}$$

$$0 = \left(R_k - \frac{2}{3}R_g\right) \circ \varepsilon_{11} + \left(2R_k + \frac{2}{3}R_g\right) \circ \varepsilon_{22} \tag{24}$$

Which yields a relation between the transverse and longitudinal strains:

$$\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{(2R_k + 2R_g/3)^{-1} \circ (R_k - 2R_g/3) \circ \varepsilon_{11}}{\varepsilon_{11}} \tag{25}$$

And also the ratio of longitudinal stress to longitudinal strain which is not displayed here.

2.2.3. Uniaxial creep and relaxation experiments

The aim is to derive the ageing linear viscoelastic "equivalents" of the Young's modulus and Poisson's ratio, by analogy with their definition in elasticity.

Uniaxial creep experiment. In the uniaxial creep experiment, for any given loading time t_0, the stress tensor evolution is prescribed as:

$$\sigma(t) = \sigma_{11}^0 \mathrm{H}(t - t_0)\underline{e}_1 \otimes \underline{e}_1 \tag{26}$$

First part of Eq. (21) then allows to define the "uniaxial compliance function" J_E:

$$J_E(t,t_0) = \frac{\varepsilon_{11}(t)}{\sigma_{11}(t)} = (J_k/9 + J_g/3)(t,t_0) \tag{27}$$

While first part of Eq. (22) then allows to define the "creep Poisson's ratio" ν_c.

$$\nu_c(t,t_0) = -\frac{\varepsilon_{22}(t)}{\varepsilon_{11}(t)} = -\frac{(J_k/9 - J_g/6)(t,t_0)}{(J_k/9 + J_g/3)(t,t_0)} \tag{28}$$

As is well-known, the creep Poisson's ratio is not constant unless the volumetric and deviatoric compliances are proportional which might be reasonable or not depending on the considered material. These Eqs. (27) and (28) are identical to equations given by Bažant (1975); Bažant and Wittmann (1982).

Recalling that in elasticity the relations between the Young's modulus and the Poisson's ratio on the one hand, and on the bulk and the shear modulus on the other hand are:

$$E = \frac{9KG}{3K+G}, \quad \nu = \frac{3K - 2G}{2(3K+G)} \tag{29}$$

it is interesting to note that Eqs. (27) and (28) can be rewritten:

$$\frac{1}{J_E(t,t_0)} = \frac{\frac{9}{J_k J_g}}{\frac{3}{J_k} + \frac{1}{J_g}}, \quad \nu_c(t,t_0) = \frac{\frac{3}{J_k} - \frac{2}{J_g}}{2(\frac{3}{J_k} + \frac{1}{J_g})} \tag{30}$$

The inverses of the compliances (but not directly the relaxations) and the CPR verify the same relations as their counterparts (in terms of inverses of stiffnesses) in elasticity. Let us also note that the partial time derivative of the CPR can be conveniently written as a function of the ratio of the shear and volumetric compliances:

$$\frac{\partial \nu_c}{\partial t}(t,t_0) = \frac{9}{2}\frac{\frac{\partial (J_g/J_k)}{\partial t}(t,t_0)}{\left(3\frac{J_g(t,t_0)}{J_k(t,t_0)} + 1\right)^2} \tag{31}$$

which, to the knowledge of the present authors, has not been pointed out before. This equation has the important consequence that the time variations of the CPR are identical to those of the ratio of the deviatoric and volumetric compliances.

The viscoelastic linear isotropic constitutive law can conveniently be rewritten using the CPR and the uniaxial compliance:

$$\varepsilon = \left(\left(\frac{1 - 2\nu_c}{3}J_E\right) \circ \operatorname{tr} \sigma\right)1 + ((1 + \nu_c)J_E) \circ \sigma^{\text{dev}} \tag{32}$$

Uniaxial relaxation experiment. In the uniaxial relaxation experiment, the evolution of the component 11 of the strain is prescribed as:

$$\varepsilon_{11}(t) = \varepsilon_{11}^0 \mathrm{H}(t - t_0) \tag{33}$$

And the components 22, 33, 23, 13, 12 of the stress tensor are constantly equal to 0. Thus, the particular case studied in Section 2.2.2 is also applicable here.

Second part of Eq. (21) then allows to define the "uniaxial relaxation function" R_E:

$$\frac{1}{R_E(t,t_0)} = \frac{\varepsilon_{11}(t)}{\sigma_{11}(t)} = \frac{1}{(J_k/9 + J_g/3)^{-1}(t,t_0)} \tag{34}$$

While second part of Eq. (22) allows to define the "relaxation Poisson's ratio" ν_r:

$$\nu_r(t,t_0) = -\frac{\varepsilon_{22}(t)}{\varepsilon_{11}(t)} = -\left((J_k/9 - J_g/6) \circ (J_k/9 + J_g/3)^{-1}\right)(t,t_0) \tag{35}$$

This coefficient has been defined in the Laplace-Carson transformed domain by Mandel (1966); Salençon (1983) (under the name relaxation Poisson's ratio) and Christensen (2012) (under the name viscoelastic Poisson's ratio), and also in the general case and in the time domain by Salençon (2009).

Using Eq. (25), the relaxation Poisson's ratio can also be written:

$$\nu_r(t,t_0) = -\frac{\varepsilon_{22}(t)}{\varepsilon_{11}(t)} = \left((2R_k + 2R_g/3)^{-1} \circ (R_k - 2R_g/3)\right)(t,t_0) \tag{36}$$

This relation is similar to a relation given by Mandel (1958).

Again, it is interesting to rearrange these equations in the same manner as the well known relations used in elasticity (29):

$$\begin{aligned} R_E &= 9R_g \circ (R_g + 3R_k)^{-1} \circ R_k \\ &= 9R_k \circ (R_g + 3R_k)^{-1} \circ R_g, \\ \nu_r &= (2(3R_k + R_g))^{-1} \circ (3R_k - 2R_g) \end{aligned} \tag{37}$$

However, note that directly deriving Eqs. (37) from the elasticity relations (29) is not possible due to the non commutativity of the Volterra integral operator.

The viscoelastic linear isotropic constitutive law can conveniently be rewritten using the relaxation Poisson's ratio and the uniaxial relaxation function:

$$\sigma = \frac{R_E}{3} \circ (\mathrm{H} - 2\nu_r)^{-1} \circ (\mathrm{tr}\,\varepsilon)1 + R_E \circ (\mathrm{H} + \nu_r)^{-1} \circ \varepsilon^{\mathrm{dev}} \tag{38}$$

2.2.4. Relation between both Poisson's ratios

Multiplying (in the sense of the Volterra integral operator) (27) by the inverse (in the sense of the product between scalars) of (34) allows to retrieve that the uniaxial compliance and relaxation functions are inverses (in the sense of the Volterra integral operator):

$$J_E \circ R_E = \mathrm{H} \tag{39}$$

Eliminating $J_k/9 - J_g/6$ from Eqs. (28) and (35), and substituting $J_k/9 + J_g/3$ by J_E, yields a relation between both Poisson's ratios:

$$\nu_c J_E = \nu_r \circ J_E \tag{40}$$

where on the left hand side the product is the usual product between scalars, and on the right hand side, the product is the Volterra integral operator. The formula linking the two Poisson's ratios can be found, for example, in Salençon (1983); Salençon (2009). As mentioned by Aili et al. (2015b), it is different from the relation derived by Lakes and Wineman in Lakes and Wineman (2006).

Table 1
A summary of useful relations in ageing linear viscoelasticity.

$J_E = J_k/9 + J_g/3$
$\nu_c = \frac{3J_g - 2J_k}{2(3J_g + J_k)}$
$J_g = 2(1+\nu_c)J_E$
$J_k = 3(1-2\nu_c)J_E$
$R_E = 9R_g \circ (R_g + 3R_k)^{-1} \circ R_k$
$\nu_r = (2(3R_k + R_g))^{-1} \circ (3R_k - 2R_g)$
$R_g = \frac{R_E}{2} \circ (\mathrm{H} + \nu_r)^{-1}$
$R_k = \frac{R_E}{3} \circ (\mathrm{H} - 2\nu_r)^{-1}$
$R_k \circ J_k = \mathrm{H}$
$R_g \circ J_g = \mathrm{H}$
$J_E \circ R_E = \mathrm{H}$
$\nu_c J_E = \nu_r \circ J_E$

2.2.5. A practical summary of relations between compliances, relaxation functions and viscoelastic Poisson's ratios in linear isotropic ageing viscoelasticity

In this section, the main relations presented in the previous sections are gathered in a compact form in Table 1 (the time dependences are omitted for clarity). Three sections are proposed: relations between the compliance functions and the CPR, relations between the relaxation functions and the RPR, relations between compliances and relaxation functions.

2.3. Isotropic non ageing viscoelasticity

2.3.1. From Stieltjes integrals in the time domain to simple products in the transformed domain

Relations given in this section for the case of non ageing viscoelasticity can be rewritten in the case of non ageing viscoelasticity, which makes the use of the Laplace–Carson transform introduced in Section 2.1.3 possible. Such equations have been known for decades. A short presentation by Mandel can be found in Mandel (1955).

First, keeping the equations in the time domain, Stieltjes integrals become convolution products: in all equations, the symbol $\circ$ can simply be replaced by $*$, denoting a Stieltjes convolution product. Then, applying the Laplace–Carson transform to the equations as done in Section 2.1.3, convolution products become simple products. Let us only apply this procedure to Eq. (37) as an example:

$$\begin{aligned} \nu_r(t,t_0) = \nu_r(t-t_0) &= (2(3R_k + R_g))^{-1} * (3R_k - 2R_g) \\ \nu_r^* &= \frac{3R_k^* - 2R_g^*}{2(3R_k^* + R_g^*)} \end{aligned} \tag{41}$$

The viscoelastic linear non ageing isotropic constitutive law can conveniently be rewritten in the transformed space using the relaxation Poisson's ratio and the uniaxial relaxation function:

$$\sigma^* = \frac{R_E^*}{3(1-2\nu_r^*)}(\mathrm{tr}\,\varepsilon^*)1 + \frac{R_E^*}{1+\nu_r^*}\varepsilon^{*\mathrm{dev}} \tag{42}$$

The equations obtained are identical to equations in elasticity, which was noted long ago by Lee, Biot, Mandel, and others (Lee, 1960; Biot, 1958; Mandel, 1955) and gave birth to the well-known correspondence principle (such a principle was stated even earlier by Read using the Fourier transform (Read, 1950), and by Sips (1951).

Table 2
A summary of useful relations in non-ageing linear viscoelasticity.

$R_E^* = \frac{9R_g^*R_k^*}{R_g^*+3R_k^*}$	$R_k^*J_k^* = 1$
$\nu_r^* = \frac{3R_k^*-2R_g^*}{2(3R_k^*+R_g^*)}$	$R_g^*J_g^* = 1$
$R_g^* = \frac{R_E^*}{2(1+\nu_r^*)}$	$J_E^*R_E^* = 1$
$R_k^* = \frac{R_E^*}{3(1-2\nu_r^*)}$	$(\nu_c J_E)^* = \nu_r^* J_E^*$

2.3.2. A practical summary of relations between compliances, relaxation functions and viscoelastic Poisson's ratios in linear isotropic non ageing viscoelasticity

In the non ageing case, the relations between the relaxation functions are gathered in Table 2.

3. Discussion

Even if the theory presented in the previous section has been known for a long time in the continuum mechanics community, some discussion is still needed on some issues. Here, the questions of the thermodynamical restrictions on the relaxation and compliance functions as well as the applicability of the viscoelastic Poisson's ratio are addressed.

3.1. Restrictions on the relaxation functions

In isotropic elasticity, it can be derived easily that all moduli (Young's, shear, bulk moduli) must be positive. As a consequence, the elastic Poisson's ratio lies between -1 and $1/2$. In the viscoelastic context, analogous relations are also useful in order to check the consistency of empirical models with thermodynamics. The aim of this section is hence to motivate the need for future work in that field.

3.1.1. Isotropic ageing linear viscoelasticity

To the knowledge of the authors, little information can be found in the literature regarding the thermodynamical restrictions on the values of the relaxation functions or of the compliance functions. Of course, the most intuitive of these relations is that the relaxation and compliance functions are positive at all times and for all loading times. Going back to Eq. (28), a consequence is that the CPR also lies between -1 and $1/2$ both in the ageing and non-ageing cases. However, one can imagine that more relations exist and should be proven thermodynamically, but the authors could not prove that the relaxation PR respects these bounds, except in the transformed domain in the non-ageing case, which is analogous to elasticity.

Bažant mentions in Bažant (1975) that all test data on concrete agree with the following inequalities:

$$\forall\ (t,t'),\ \frac{\partial J_E(t,t')}{\partial t} \geq 0,\ \ \frac{\partial^2 J_E(t,t')}{\partial t^2} \leq 0 \tag{43}$$

$$\frac{\partial J_E(t,t')}{\partial t'} \leq 0,\ \ \frac{\partial^2 J_E(t,t')}{\partial t'^2} \geq 0 \tag{44}$$

but unfortunately no mathematical proof is proposed for these intuitive relations. Eq. (43) mean that the compliance functions are increasing functions of time and that the rate of increase decreases over time. Eq. (44) mean that when looking at strains at a given time in a creep tests, these strains are smaller if loading has been performed later, to an extent which is greater if the loading time is larger.

Bažant gives other intuitive relations of the same kind in Bažant and Wittmann (1982). In addition to Eqs. (43) and (44), the following expression is added:

$$\left[\frac{\partial J_E(t,t')}{\partial t'}\right]_{t-t'} \leq 0 \tag{45}$$

which means that looking at strains at a given amount of time after loading, these strains are lower if loading was performed later. In our opinion, this relation which seems well adapted to concrete cannot be general, since it depends on the kind of ageing undergone by the material. The equation:

$$\forall t' > t,\ \ \frac{\partial^2 J(t,t')}{\partial t \partial t'} \geq 0 \tag{46}$$

is also given, and is explained as a condition of non-divergence of creep curves (corresponding to different loading times).

Jiràsek has recently studied compliances functions from various creep models for concrete (Jiràsek, 2015). The criterion used has been the monotonicity of the recovery curves (i.e. strain after unloading of a creep test). This condition translates on the compliance function as Eq. (46), even if the physical meaning is slightly different since it involves recovery curves.

As a conclusion, the present authors have not been able to find a detailed presentation of the thermodynamical restrictions on compliance and relaxation functions in the general ageing case.

3.1.2. Isotropic non ageing linear viscoelasticity

Biot has first proposed a thermodynamic analysis of the linear viscoelasticity (Biot, 1954), based on Onsager's principle. He showed that the relaxation function can be written using a relaxation spectrum, and that all characteristic times were positive, in the context of anisotropic linear viscoelasticity. This analysis was further developed by Mandel (1966), showing that the spectra of the relaxation functions and compliances are non-negative (in the tensorial case). As a consequence, the relaxation functions $R(t)$ and compliances $J(t)$ have alternate derivatives:

$$(-1)^n R^{(n)} \geq 0, \text{ and } (-1)^n J^{(n)} \leq 0,\ \ n \geq 1 \tag{47}$$

In the multiaxial isotropic case, these relations apply to the eigenvalues of the compliance of relaxation tensors, which are J_k and J_g for the compliance tensor, and R_k and R_g for the relaxation tensor in the isotropic case.

Also from a thermodynamics point of view, but avoiding the use of Onsager's principle, Christensen derived restrictions on shear and bulk relaxation functions in the non ageing case (Christensen, 1972; 2012). The requirement that the stored energy under a strain step is non-negative yields:

$$R_k(t) \geq 0,\ \ R_g(t) \geq 0 \tag{48}$$

Moreover, the requirement that the rate of energy dissipation must be non-negative yields:

$$R'_k(t) \leq 0,\ \ R'_g(t) \leq 0 \tag{49}$$

Finally, the so-called condition of fading memory yields:

$$R''_k(t) \geq 0,\ \ R''_g(t) \geq 0 \tag{50}$$

However, Christensen acknowledges a condition of alternate signs of all derivatives of the relaxation functions close to Mandel (1966) proposed by Day (1970), but mentions it is not possible to prove these conditions from the three principles mentioned earlier (Christensen, 1972). Christensen does not acknowledge Mandel's work, perhaps due to the fact that it was published in French.

Concerning the CPR, using relation for ν_c given in Eq. (30), and the fact that deviatoric and spherical compliances are positive, it can be shown, as in elasticity, that the CPR lies between -1 and $1/2$. The same can be easily shown for the RPR in the transformed domain. All RPR computed (in ageing or non ageing case) in the time domain lied between the same boundaries, but no proof has been found to show that this is general.

As a conclusion, more work is needed to improve the knowledge on the thermodynamical restrictions on viscoelastic parameters.

3.2. Creep and relaxation Poisson's ratios

3.2.1. Variation of the viscoelastic Poisson's ratios

As shown in Eq. (31), the partial time derivative of the CPR can be related to the time derivative of the ratio J_g/J_k, and has the same sign. This makes clearer the common sense that if creep is faster in shear, the CPR increases, while if it is faster in volume, it decreases. In fact it is not the ratio of the derivatives of the compliances which is concerned, but the derivative of the ratio of the compliances. Therefore, if the ratio J_g/J_k decreases, the CPR also decreases. However, the authors were not able to derive a similar general expression on the variation of the RPR, which would have been useful to verify Tschoegl's statement (Tschoegl et al., 2002) that the viscoelastic Poisson's ratio (corresponding to our relaxation Poisson's ratio) is a non-decreasing function of time. Instead of that, practical examples will be shown in Section 4 both in ageing and non ageing viscoelasticity in contradiction with that statement which was already pointed to be wrong by Lakes and Wineman (2006).

3.2.2. Discussion about the correspondence principle and viscoelastic Poisson's ratio

It has been argued in the literature that the elastic/viscoelastic analogy does not apply to the viscoelastic Poisson's ratio (Hilton and Yi, 1998). It has been shown here that one just needs to be careful so as to correctly define the Poisson's ratio in viscoelasticity. If one deals with the creep Poisson's ratio, which is commonly used by experimental researchers, particularly in the field of concrete science since it is straightforward to compute from experiments, the correspondence principle cannot be used. However, it can be used with the relaxation Poisson's ratio, which can be easily computed if the creep Poisson's ratio and the uniaxial compliance are known (the numerical inversion of the integral equation can be done following (Bažant, 1972; Sorvari and Malinen, 2007).

As mentioned earlier, due to the validity of the correspondence principle for the relaxation Poisson's ratio, some authors have used it to define the relaxation Poisson's ratio directly in the transformed domain (Salençon, 1983; Christensen, 2012; Lakes, 1992; Lakes and Wineman, 2006; Tschoegl et al., 2002; Grasley and Lange, 2007).

3.2.3. Comparison with Hilton's classification of viscoelastic Poisson's ratios

It has been argued by Hilton and Yi (1998), amongst others, that the viscoelastic Poisson's ratio is load-history dependent, and that this fact prevents from using these functions to describe the material behaviour in general cases. Here, following early works on linear viscoelasticity, it has been shown that although the viscoelastic Poisson's ratio is load-history dependent, it is perfectly consistent to accept this fact and define two particular cases of viscoelastic Poisson's ratio which were called relaxation and creep Poisson's ratio since they are equal to the opposite ratio of lateral to axial strains in experiments of the same name. Both these coefficients can be used in the time domain to describe generally the behaviour of any linear isotropic viscoelastic solid. However, it has also been shown that only the relaxation Poisson's ratio can be used in the transformed domain in the non ageing case.

It is interesting to note that the two viscoelastic Poisson's ratio defined here are not perfectly consistent with the five classes of Poisson's ratios defined by Hilton (2001, 2009). Class I corresponds to the opposite of the ratio of lateral to axial strains, without further precisions about the loading. Class II is based on the same equation, except that the axial strain is constant, without further information about strains or stresses in the other directions. Class III is based on Fourier transform but again the loading is not specified. Class IV and V are based on the Hencky strain and on the strain rates respectively. Therefore, the definition used in this paper both belong to Class I, for two different particular loadings corresponding to the uniaxial creep and relaxation experiments. Some additional classes of viscoelastic Poisson's ratios are defined in Hilton (2011), but again the distinction between creep and relaxation Poisson's ratio (both belonging to class I) is not made, and the relation between these coefficients is not given.

Finally, both Poisson's ratio are valid material parameters, each having advantages or disadvantages depending on the application. However, the use of spherical and deviatoric compliances or relaxation functions might be preferable because less risks of error exist.

4. Applications to concrete

In order to illustrate how the CPR and RPR can be computed and how they evolve in time, different applications are proposed, all belonging to concrete science.

4.1. Non ageing viscoelasticity: Maxwell matrix and elastic inclusions

A composite material made up of elastic inclusions embedded into a Maxwell non ageing viscoelastic matrix is first considered. To focus on the influence of the mechanical interactions between inclusions and matrix, the elastic and viscous Poisson's ratios of matrix are taken as equal. The Poisson's ratio (either the creep or the relaxation one) of matrix is thus constant. The matrix relaxation tensor reads:

$$\mathbb{R}_m(t,t') = E_m \mathrm{e}^{-\frac{t-t'}{\tau_m}} \mathrm{H}(t-t') \left(\frac{1}{1-2\nu_m} \mathbb{J} + \frac{1}{1+\nu_m} \mathbb{K} \right) \tag{51}$$

where E_m is the elastic Young's modulus of matrix, ν_m is the Poisson's ratio, τ_m is the Maxwell characteristic time. Inclusions are elastic, characterised by the Young's modulus E_i and the Poisson's ratio ν_i. The volume fraction of inclusions is denoted by f_i. The effective behaviour of this matrix-inclusions composite material is estimated using the Mori–Tanaka scheme (Mori and Tanaka, 1973). The bulk and shear effective behaviours are found to correspond to generalized Maxwell models (Ricaud and Masson, 2009), with two Maxwell chains. The expressions of the properties of these generalized Maxwell models are too lengthy to be reproduced here.

The effective creep and relaxation Poisson's ratios are plotted on Fig. 1 for two values of the inclusion to matrix elastic contrast, either softer ($E_i/E_m = 0.1$) or stiffer ($E_i/E_m = 10$) than the matrix. Clearly enough, even if both Poisson's ratios of matrix and inclusions are constant (and taken as equal in this first application), the effective Poisson's ratios are not constant, due to mechanical interactions between phases. Furthermore, the effective Poisson's ratios can be non monotonic (case $E_i/E_m = 0.1$), which is in contradiction with Tschoegl statement that viscous Poisson's ratio are non-decreasing (Tschoegl et al., 2002). On this example, the stiffness contrast between inclusions and matrix is found to have a varying influence (as also evidenced by Aili et al. (2015b), depending on time, on the effective Poisson's ratios:

- at $t \to 0$ (elastic behaviour), influence of contrast is moderate,
- at finite times, influence of contrast is greater than in the elastic case,
- at $t \to \infty$, contrast does not have any influence as only the matrix viscous behaviour influences the effective Poisson's ratios.

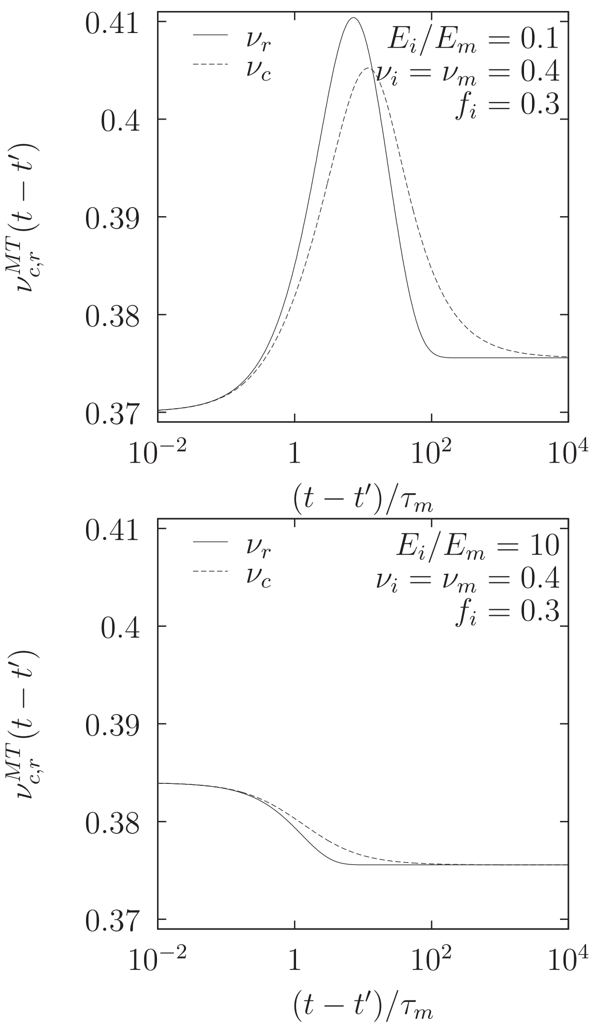

Fig. 1. Non ageing Maxwell matrix and elastic inclusions: effective relaxation and creep Poisson's ratios, influence of elastic contrast between inclusions and matrix.

The influence of the inclusions volume fraction is considered on Fig. 2. Depending on the volume fraction, the effective Poisson's ratios can be non monotonic. Furthermore, the relaxation Poisson's ratio can be non monotonic while the creep one is strictly monotonic (case $f_i = 0.5$).

The effective behaviour having the structure of a generalized Maxwell model with two chains (with different parameters in bulk and shear), note that various evolutions (non-monotonic, increasing, decreasing) of Poisson's ratios are possible even with a simple (only 2 chains) rheologic model. It is even possible to encounter simultaneously a monotonic creep Poisson's ratio and a non monotonic relaxation Poisson's ratio. As can be easily shown (see Aili et al., 2015b), the initial and final values of the creep and relaxation Poisson's ratios are identical in the non ageing context.

4.2. Ageing viscoelasticity: Bažant solidification theory

To extend the non ageing analysis proposed by Aili et al. (2015b), a first application to ageing viscoelastic behaviours is considered, using Bažant solidification theory (Bažant, 1977) to define the bulk and shear behaviours. In this theory, here extended to relaxation tensors, a non ageing relaxation tensor $\mathbb{R}^{na}$ is multiplied by a so-called ageing function f^a depending on loading time t':

$$\mathbb{R}(t,t') = f^a(t')\mathbb{R}^{na}(t-t') \quad (52)$$

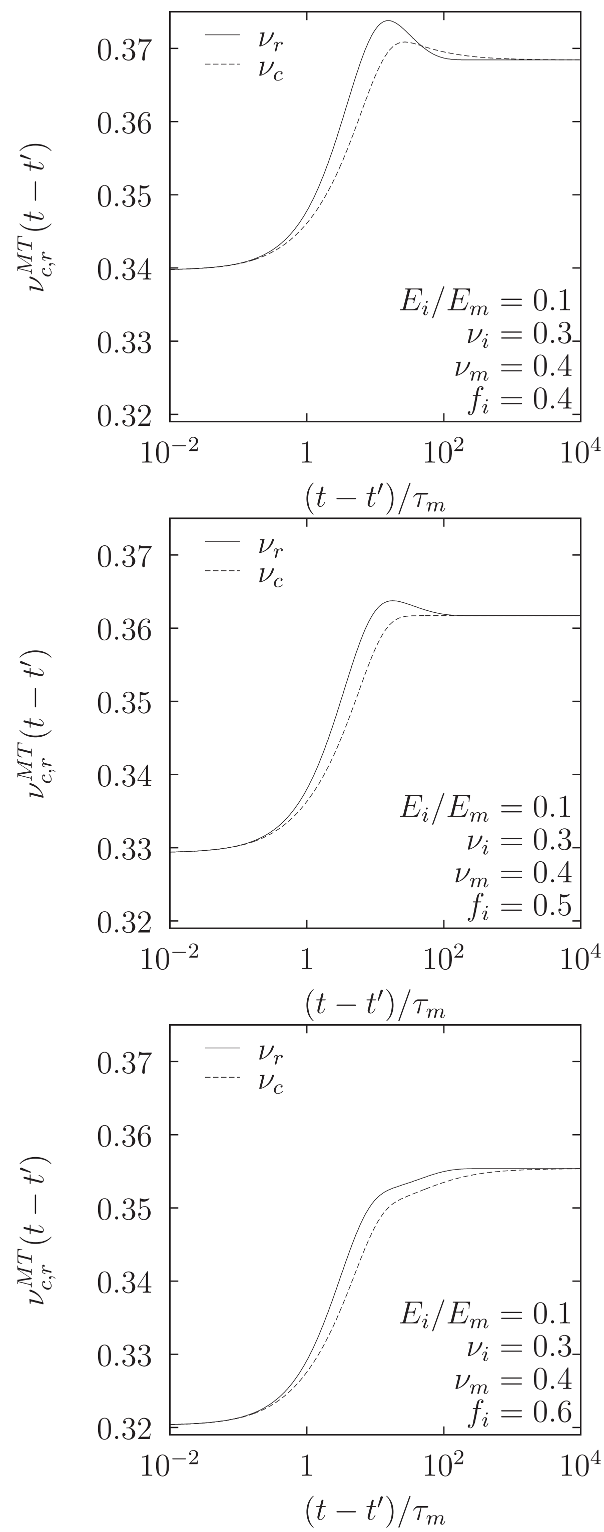

Fig. 2. Non ageing Maxwell matrix and elastic inclusions: effective relaxation and creep Poisson's ratios, influence of inclusions volume fraction.

For the sake of simplicity, the non ageing behaviour is here represented by an isotropic Maxwell model. Spherical and deviatoric viscoelastic properties (stiffness and viscosity) are assumed to be different:

$$\mathbb{R}^{na}(t-t') = \left(3k e^{-\frac{t-t'}{\eta/k}}\mathbb{J} + 2\,g e^{-\frac{t-t'}{\gamma/g}}\mathbb{K}\right)\mathrm{H}(t-t') \quad (53)$$

Table 3
Material parameters used in application.

Non ageing behaviour		Ageing function		
ν^e	ν^v	f^{a0}	$f^{a\infty}$	τ^a/τ
0.4	0.1	0.1	1	2

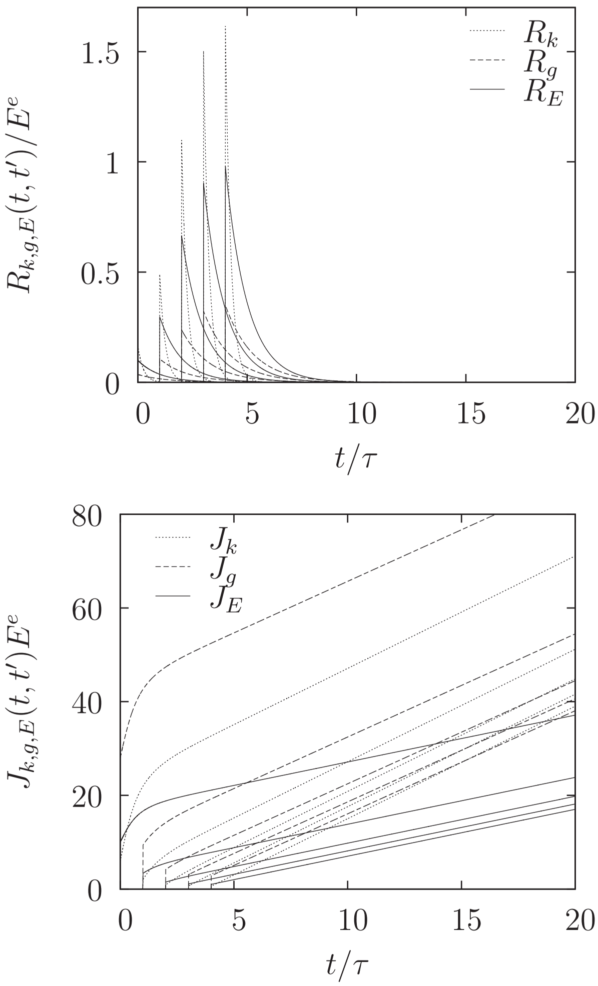

Fig. 3. Relaxation and creep functions for $t'/\tau = 0, 1, 2, 3, 4$.

Alternatively, the elastic part (springs) of the isotropic Maxwell behaviour can be defined introducing the elastic Young's modulus E^e and Poisson's ratio ν^e, using the classical relations:

$$k = \frac{E^e}{3(1-2\nu^e)} \quad \text{and} \quad g = \frac{E^e}{2(1+\nu^e)} \tag{54}$$

And the viscous part (dashpots) can be defined, by extension, using the viscous Young's modulus E^v and Poisson's ratio ν^v:

$$\eta = \frac{E^v}{3(1-2\nu^v)} \quad \text{and} \quad \gamma = \frac{E^v}{2(1+\nu^v)} \tag{55}$$

The elastic Young's modulus E^e and the Maxwell uniaxial characteristic time $\tau = E^v/E^e$ are used to adimensionalize stresses and time.

The ageing function is taken as:

$$f^a(t') = f^{a0} + (f^{a\infty} - f^{a0})\left(1 - \mathrm{e}^{-(t'/\tau^a)^2}\right) \tag{56}$$

where f^{a0} and $f^{a\infty}$ are respectively the initial and final values, and τ^a is the ageing characteristic time.

The (arbitrary chosen) material parameters are gathered in Table 3.

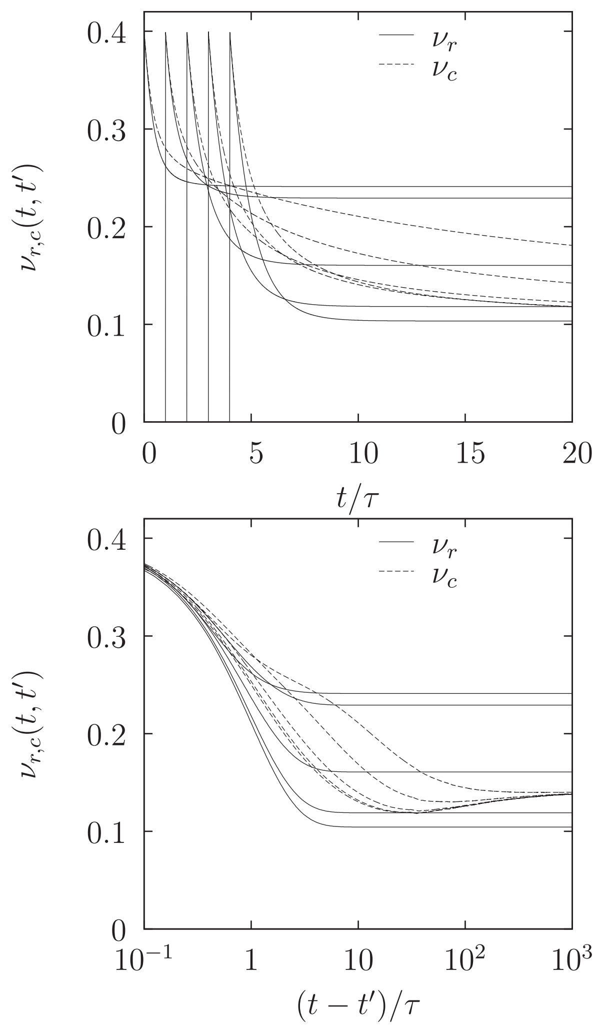

Fig. 4. Relaxation and creep Poisson's ratios for $t'/\tau = 0, 1, 2, 3, 4$ (left: linear time scale w.r.t t; right: logarithmic time scale w.r.t $t - t'$).

Bulk, shear and uniaxial relaxation and creep functions are plotted on Fig. 3: the behaviour is clearly ageing. Relaxation and creep Poisson's ratios are plotted on Fig. 4. While both Poisson's ratios have the same initial value (at $t \to t'$) and tangent, the final values (when $t \to \infty$) are found to differ, contrary to the non ageing case. Furthermore, in this application, the creep Poisson's ratio seems to converge when $t \to \infty$ towards a unique value irrespective of the loading time t', while it is not the case for the relaxation Poisson's ratio. And the latter reaches much faster its asymptotic value than the creep Poisson's ratio.

4.3. Multiscale evolution of the CPR using Vi(CA)$_2$T

In order to illustrate how creep Poisson's ratio can evolve in time, an application to the modeling of the mechanical properties of concrete is proposed. For more details on this example, the reader can refer to Charpin et al. (2016). A software dedicated to the prediction of the properties of concrete has been developed at EDF R&D since 2006. It is called Vi(CA)$_2$T for Virtual Concrete And Cement Ageing Analysis Toolbox. A detailed description of the software can be found in Sanahuja et al. (2016a, 2016b). The basic information needed as input by the software is the mix of concrete, the properties of the cement and aggregates used, as well as the mechanical and physical properties of the hydrates. First, a hydration module computes the evolution of the volume fraction

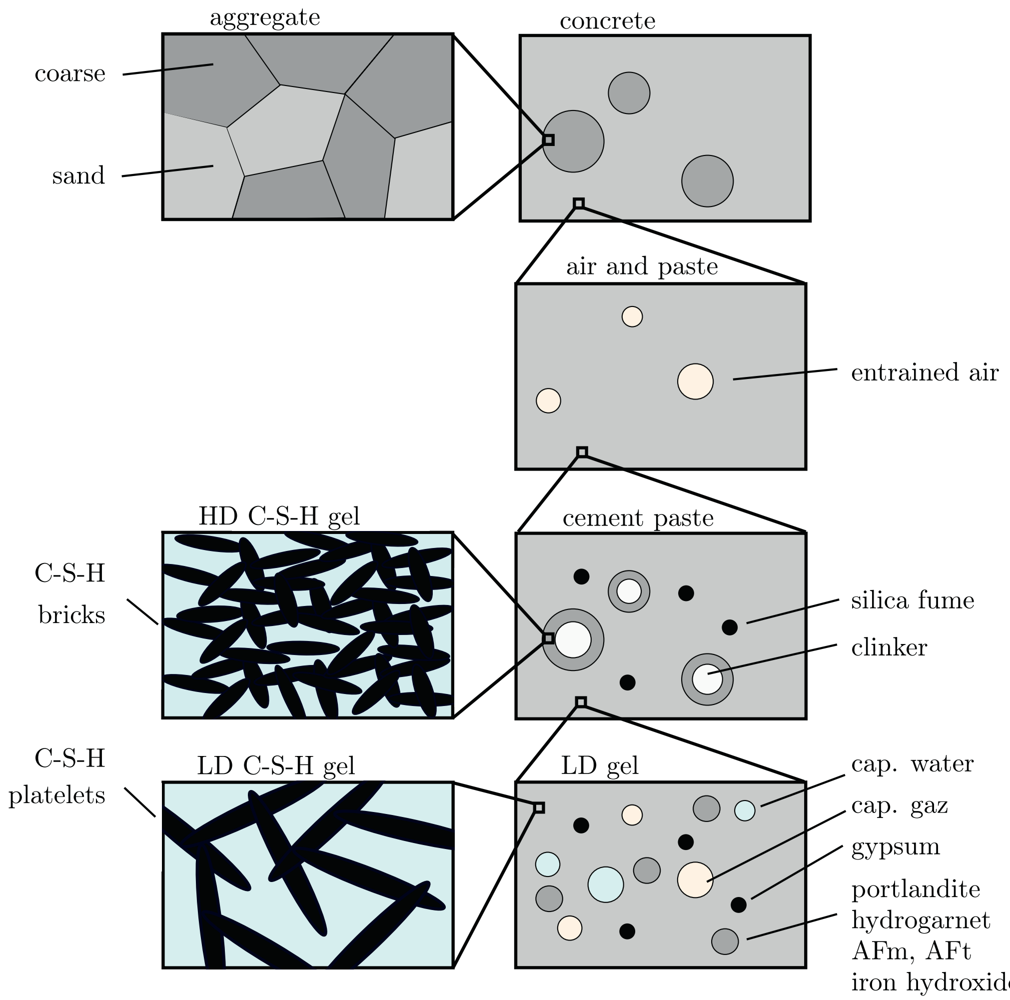

Fig. 5. Multiscale morphology of concrete used in Vi(CA)$_2$T V2.1.2.

of the various phases of concrete (anhydrous, water and hydrates). Second, a morphological model describes the multiscale arrangement of those phases in space as well as the shapes of the inclusion phases. The morphological model used in the current version of Vi(CA)$_2$T is displayed on Fig. 5.

Finally, mechanical models based on micromechanics predict elasticity (based on Sanahuja et al., 2007) and basic creep on frozen microstructures thanks to recent developments based on Sanahuja et al. (2011); Sanahuja and Dormieux (2010). Since the microstructure evolves, the creep response depends on the time of loading. However, once the loading is applied, the microstructure remains frozen. Therefore, the Laplace transform can be used to determine the time response for a loading at a given time using the corresponding elasticity problem in the transformed domain. The elementary creep mechanism is a sliding mechanism at the level of C-S-H bricks and platelets. These solid particles are represented by oblate spheroids. Their instantaneous mechanical behaviour is isotropic, but their delayed behaviour is not, since creep only occurs in their plane. This creep behaviour is described by a Maxwell model, which adds only one parameter compared to elasticity. All other phases are elastic. Therefore, at the lowest level, creep is only deviatoric. When moving to larger scales, spherical creep appears due to the presence of voids and rigid inclusions, which will be shown in this example.

The uniaxial and transverse strains in a fictitious creep test are computed at all levels of the microstructure. The creep Poisson's

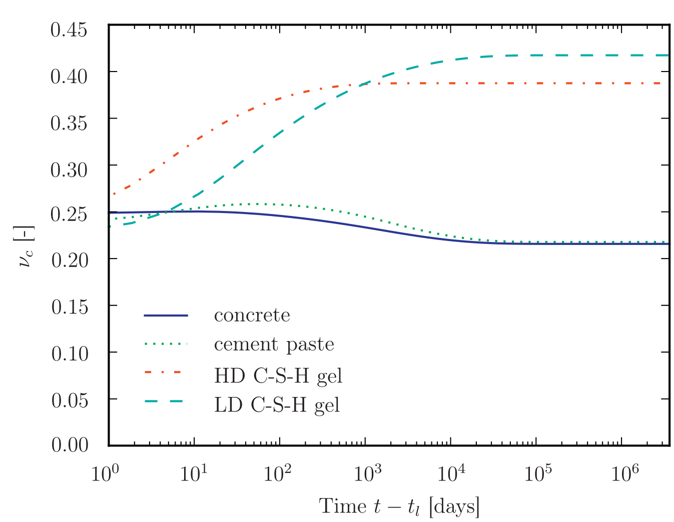

Fig. 6. Creep Poisson's ratios at the main levels of the morphological scale. Loading at 90 days.

ratio is very high at the lowest scales (C-S-H gels), which means that creep is almost completely deviatoric (Fig. 6). This consistent with the fact that the elementary creep mechanism is totally deviatoric, and the fact that these gels are formed as a packing of

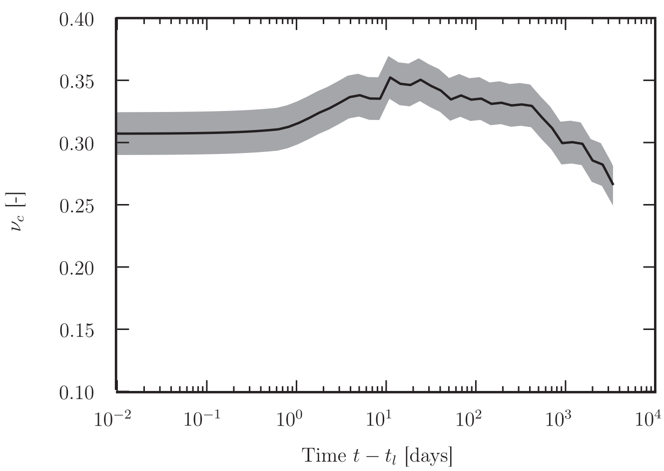

Fig. 7. Creep Poisson's ratio computed from basic creep biaxial tests performed at EDF. The shaded area represents the uncertainty related to the thermal dilation of the sample due to the temperature variations in the testing room.

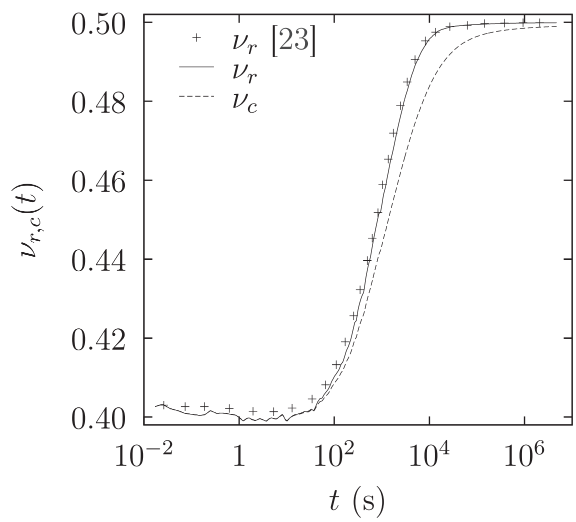

Fig. 8. Relaxation and creep Poisson's ratios of polycarbonate, computed from data in Grassia et al. (2010).

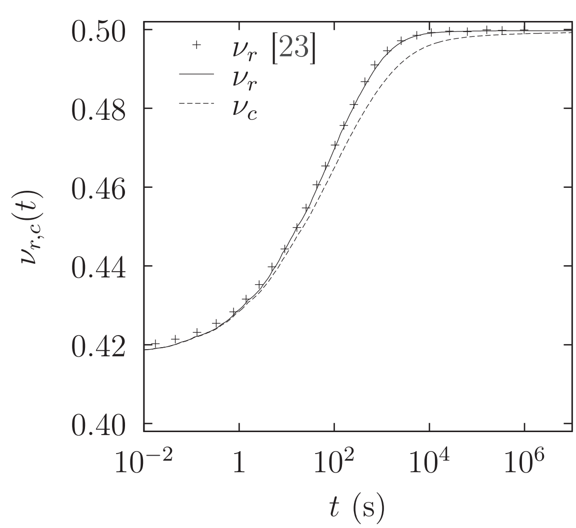

Fig. 9. Relaxation and creep Poisson's ratios of polycyanurate, $x_M = 0.1$, computed from data in Grassia et al. (2010).

C-S-H elementary bricks. At larger scales, due to the incorporation of porosity and rigid inclusions, a larger part of creep occurs under spherical loading, which induces a much lower CPR which in turn, remains almost constant during the creep test.

This example shows that the variations of the CPR can be very diverse, including non-monotonous, as is the CPR of hardened cement paste on Fig. 6.

4.4. Computation of the creep Poisson's ratio for biaxial tests

As a final application to the use of the theory of isotropic linear viscoelasticity to compute CPRs of concrete, an experimental illustration is proposed. Biaxial creep tests were started at EDF in 2004 in order to gain a better knowledge of the multiaxial behaviour of concrete in concrete containment buildings of nuclear power plants. These tests have been described in Charpin et al. (2015) and will be the focus of a detailed article (Charpin et al., To be submitted).

Under a biaxial state of stress, where stresses are applied in the vertical and horizontal directions, leaving the third direction unloaded, a direct application of Eq. (32) yields the following expression for the CPR:

$$\nu_c(t) = \frac{\sigma_h \varepsilon_v(t) - \sigma_v \varepsilon_h(t)}{\sigma_v \varepsilon_v(t) - \sigma_h \varepsilon_h(t)} \quad (57)$$

This coefficient has been computed for basic creep (using the difference of the strains measured in the basic creep test and in the autogenous shrinkage test) and is shown on Fig. 7. The CPR computed from biaxial tests is almost constant.

5. Application to amorphous polymers

Polymers also exhibit a viscoelastic mechanical behaviour. Grassia et al. (2010) have collected from the literature the bulk and shear behaviours of several amorphous polymers. They have computed the relaxation Poisson's ratio considering the material as non ageing, taking advantage of the Laplace transform, from the same equation as (41). From this experimental evidence, the relaxation Poisson's ratio is found to be non monotonic for one polymer (polycarbonate).

In this section, the bulk and shear compliance or relaxation functions reported by Grassia et al. (2010) are reused to compute the relaxation Poisson's ratio using a different technique: it is directly computed in the time domain from relaxation functions using the expression given in Table 1 or Eq. (36). The creep Poisson's ratio is also computed directly in the time domain from the compliance functions. These functions are obtained by a numerical inversion of the equations of the last block in Table 1 or Eq. (28). Results are plotted on Fig. 8 for polycarbonate and Fig. 9 for polycyanurate, $x_M = 0.1$ (the latter being the mole fraction of monofunctional monomer used in the material preparation). The relaxation Poisson's ratio obtained by Grassia et al. (2010) is plotted as dots. The relaxation Poisson's ratios are consistent, up to the fact that bulk and shear compliance or relaxation functions have been manually digitized on Grassia et al. (2010), yielding some noise especially at lower times. The creep Poisson's ratios are found to be slightly lower compared to the relaxation ones, as also numerically evidenced in Section 4.1 for cases where the Poisson's ratio are increasing.

The advantage of the computation of the CPR proposed here is that it only requires the inversion of relaxation functions to compliances, which was performed by a numerical discretization of the inverse relations. Therefore, no direct or inverse Laplace transform was needed. One can also note that the relation (31) is available on the CPR. This relation could have brought interesting information in the discussion about the variations of the viscoelastic Poisson's ratio depending on those of the compliances dealt with in Grassia et al. (2010).

6. Conclusion

Several definitions of viscoelastic Poisson's ratio have been introduced by various scientific communities, notably in concrete and polymer fields. This paper proposes rederivations of both the creep and relaxation Poisson ratios using classical integral expressions of the linear viscoelastic behaviour. Both the ageing and non ageing cases are considered. Practical relations between isotropic linear viscoelastic characteristics (relaxation and compliance functions, Poisson's ratios) are gathered, similarly to classical relations between isotropic elastic characteristics. Eventually, several examples, both theoretical and practical, about concrete and polymers, show that the evolution of both Poisson's ratios can be non monotonic and quite different.

Restrictions on viscoelastic characteristics have been studied in a rather large extent in the non ageing case. However, up to our knowledge, the literature still lacks such comprehensive analyses in the ageing case. Inequalities can be found but they are more often based on intuition than on thermodynamics, and having particular materials in mind, such as concrete. In the same line of thought, restrictions on the viscoelastic Poisson's ratios are not clear: depending on the author, the latter can be either monotonic or non monotonic. Here, the possible non monotonicity has been shown resorting to examples. Even if the creep Poisson's ratio can be straigthforwardly bounded by -1 and $1/2$ as in elasticity, these bounds are only verified on examples as far as the relaxation Poisson's ratio is concerned.

Acknowledgements

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